

From interpretation of the three classical mechanics actions to the wave function in quantum mechanics

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Abstract

First, we show that there exists in classical mechanics three actions corresponding to different boundary conditions: two well-known actions, the Euler-Lagrange classical action $S_{cl}(\mathbf{x}, t; \mathbf{x}_0)$, which links the initial position \mathbf{x}_0 and its position \mathbf{x} at time t , the Hamilton-Jacobi action $S(\mathbf{x}, t)$, which links a family of particles of initial action $S_0(\mathbf{x})$ to their various positions \mathbf{x} at time t , and a new action, the deterministic action $S(\mathbf{x}, t; \mathbf{x}_0, \mathbf{v}_0)$, which links a particle in initial position \mathbf{x}_0 and initial velocity \mathbf{v}_0 to its position \mathbf{x} at time t . Mathematically, the Euler-Lagrange action can be considered as the elementary solution to the Hamilton-Jacobi equation in a new branch of non-linear mathematics, the Minplus analysis. We study, in the semi-classical approximation, the convergence of the quantum density and the quantum action, solutions to the Madelung equations, when the Planck constant \hbar tends to 0. We find two different solutions which depend on the initial density. In the first case, where the initial quantum density is a classical density $\rho_0(\mathbf{x})$, the quantum density and the quantum action converge to a classical action and a classical density which satisfy the statistical Hamilton-Jacobi equations. These are the equations of a set of classical particles whose initial positions are known only by the density $\rho_0(\mathbf{x})$. In the second case where initial density converges to a Dirac density, the density converges to the Dirac function and the quantum action converges to a deterministic action. Therefore we introduce into classical mechanics non-discerned particles, which satisfy the statistical Hamilton-Jacobi-equations and explain the Gibbs paradox, and discerned particles, which satisfy the deterministic Hamilton-Jacobi equations. When the semi-classical approximation is not valid, we conclude that the Schrödinger equation cannot give a deterministic interpretation and the statistical Born interpretation is the only valid one. Finally, we propose an interpretation of the Schrödinger wave function that depends on the initial conditions (preparation). This double interpretation seems to be the interpretation of Louis de Broglie's "double solution" idea.

I. INTRODUCTION

The aim of this paper is to show how the interpretation of the wave function in quantum mechanics can be deduced from the interpretation of the action in classical mechanics and from the study of the convergence QM-CM when the Planck constant tends to 0. First, in section 2, we show that there exist in classical mechanics three actions corresponding to different boundary conditions: two well-known actions, the Euler-Lagrange action and the Hamilton-Jacobi action, and a new action, the deterministic action. We introduce these three actions and present the fundamental relation between the Hamilton-Jacobi and Euler-Lagrange actions. Second, in section 3, we present a new branch of non-linear mathematics, the Minplus analysis that we have developed following Maslov. In this new analysis, the Hamilton-Jacobi equation can be considered as linear. Third, in section 4, we present in the semiclassical case approximation, the QM-CM convergence when the Planck constant tends to 0. It is necessary to introduce two cases: the statistical semi-classical case and the deterministic semi-classical case. Fourth, in section 5, we discuss the case where the semi-classical approximation is not valid. Finally, we propose a realistic interpretation of quantum mechanics, which is a synthesis of the three interpretations of the founding fathers of quantum mechanics at the Solvay congress in 1927: the de Broglie interpretation, the Schrödinger interpretation and the Copenhagen interpretation.

II. THE THREE CLASSICAL MECHANICS ACTIONS

Let us consider a system evolving from the position \mathbf{x}_0 at initial time $t_0 = 0$ to the position \mathbf{x} at time t where the variable of control $\mathbf{u}(s)$ is the velocity:

$$\frac{d\mathbf{x}(s)}{ds} = \mathbf{u}(s) \quad \text{for } s \in [0, t] \quad (1)$$

$$\mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{x}(t) = \mathbf{x}. \quad (2)$$

If $L(\mathbf{x}, \dot{\mathbf{x}}, t)$ is the Lagrangian of the system, when the two positions \mathbf{x}_0 and \mathbf{x} are given, the *Euler-Lagrange action* $S_{cl}(\mathbf{x}, t; \mathbf{x}_0)$ is the function defined by:

$$S_{cl}(\mathbf{x}, t; \mathbf{x}_0) = \min_{\mathbf{u}(s), 0 \leq s \leq t} \left\{ \int_0^t L(\mathbf{x}(s), \mathbf{u}(s), s) ds \right\}, \quad (3)$$

where the minimum (or more generally an extremum) is taken on the controls $\mathbf{u}(s)$, $s \in [0, t]$, with the state $\mathbf{x}(s)$ given by the equations (1)(2). The solution $(\tilde{\mathbf{u}}(s), \tilde{\mathbf{x}}(s))$ of (3)

satisfies the Euler-Lagrange equations on the interval $[0, t]$:

$$\frac{d}{ds} \frac{\partial L}{\partial \dot{\mathbf{x}}}(\mathbf{x}(s), \dot{\mathbf{x}}(s), s) - \frac{\partial L}{\partial \mathbf{x}}(\mathbf{x}(s), \dot{\mathbf{x}}(s), s) = 0 \quad (0 \leq s \leq t) \quad (4)$$

$$\mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{x}(t) = \mathbf{x}. \quad (5)$$

If $L(\mathbf{x}, \dot{\mathbf{x}}, t) = \frac{1}{2}m\dot{\mathbf{x}}^2 + \mathbf{K} \cdot \mathbf{x}$, then the Euler-Lagrange action is $S_{cl}(\mathbf{x}, t; \mathbf{x}_0) = m\frac{(\mathbf{x}-\mathbf{x}_0)^2}{2t} + \frac{\mathbf{K} \cdot (\mathbf{x}+\mathbf{x}_0)}{2}t - \frac{K^2}{24m}t^3$ and the initial velocity is given by $\mathbf{v}_0 = \dot{\mathbf{x}}(0) = -\frac{1}{m}\frac{\partial S_{cl}}{\partial \mathbf{x}_0}(\mathbf{x}, t; \mathbf{x}_0) = \frac{\mathbf{x}-\mathbf{x}_0}{t} - \frac{\mathbf{K}t}{2m}$.

Let us now consider that an initial action $S_0(\mathbf{x})$ is given, then *the Hamilton-Jacobi action* $S(\mathbf{x}, t)$ is the function defined by:

$$S(\mathbf{x}, t) = \min_{\mathbf{x}_0; \mathbf{u}(s), 0 \leq s \leq t} \left\{ S_0(\mathbf{x}_0) + \int_0^t L(\mathbf{x}(s), \mathbf{u}(s), s) ds \right\} \quad (6)$$

where the minimum is taken on all initial positions \mathbf{x}_0 , on the controls $\mathbf{u}(s)$, $s \in [0, t]$, with the state $\mathbf{x}(s)$ given by the equations (1)(2). Because the term $S_0(\mathbf{x}_0)$ has no effect in equation (6) for the minimization on the control $\mathbf{u}(s)$, we deduce the important relation between the Hamilton-Jacobi action and Euler-Lagrange action:

$$S(\mathbf{x}, t) = \min_{\mathbf{x}_0} (S_0(\mathbf{x}_0) + S_{cl}(\mathbf{x}, t; \mathbf{x}_0)). \quad (7)$$

This equation is similar to the Hopf-Lax or Lax-Oleinik formula¹.

If $L(\mathbf{x}, \dot{\mathbf{x}}, t) = \frac{1}{2}m\dot{\mathbf{x}}^2 + \mathbf{K} \cdot \mathbf{x}$ with the initial action $S_0(\mathbf{x}) = m\mathbf{v}_0 \cdot \mathbf{x}$, then the Hamilton-Jacobi action is equal to $S(\mathbf{x}, t) = m\mathbf{v}_0 \cdot \mathbf{x} - \frac{1}{2}m\mathbf{v}_0^2 t + \mathbf{K} \cdot \mathbf{x}t - \frac{1}{2}\mathbf{K} \cdot \mathbf{v}_0 t^2 - \frac{\mathbf{K}^2 t^3}{6m}$.

For a non-relativistic particle with the Lagrangian $L(\mathbf{x}, \dot{\mathbf{x}}, t) = \frac{1}{2}m\dot{\mathbf{x}}^2 - V(\mathbf{x}, t)$, we obtain the well-known result: *The velocity of a non-relativistic classical particle in a potential field is given for each point (\mathbf{x}, t) by:*

$$\mathbf{v}(\mathbf{x}, t) = \frac{\nabla S(\mathbf{x}, t)}{m} \quad (8)$$

where $S(\mathbf{x}, t)$ is the Hamilton-Jacobi action, a solution to the Hamilton-Jacobi equations:

$$\frac{\partial S(\mathbf{x}, t)}{\partial t} + \frac{1}{2m}(\nabla S(\mathbf{x}, t))^2 + V(\mathbf{x}, t) = 0 \quad (9)$$

$$S(\mathbf{x}, 0) = S_0(\mathbf{x}). \quad (10)$$

The Hamilton-Jacobi action corresponds to a velocity field $\mathbf{v}(\mathbf{x}, t) = \frac{\nabla S(\mathbf{x}, t)}{m}$. The Hamilton-Jacobi action $S(\mathbf{x}, t)$ does not solve only a given problem with a single initial condition $\left(\mathbf{x}_0, \frac{\nabla S_0(\mathbf{x}_0)}{m}\right)$, but a set of problems with an infinity of initial conditions $\left(\mathbf{y}, \frac{\nabla S_0(\mathbf{y})}{m}\right)$. It is the problem solved by Nature with the principle of least action.

In the absence of an initial velocity field as in the Hamilton-Jacobi action, the Euler-Lagrange action answers a problem posed by the observer, and not by Nature: "If we see that a particle in \mathbf{x}_0 at the initial time arrives in \mathbf{x} at time t , what was its initial velocity \mathbf{v}_0 ?"

Let us now consider that we know the initial conditions $(\mathbf{x}_0, \mathbf{v}_0)$ and the Lagrangian of the system. If $\xi(t)$ is the classical trajectory in the field $V(\mathbf{x}, t)$ of the particle with the initial conditions $(\mathbf{x}_0, \mathbf{v}_0)$, then we define *the deterministic action* $S(\mathbf{x}, t; \mathbf{x}_0, \mathbf{v}_0)$ by the equation:

$$S(\mathbf{x}, t; \mathbf{x}_0, \mathbf{v}_0) = m \frac{d\xi(t)}{dt} \cdot \mathbf{x} + g(t) \quad (11)$$

where $g(t) = - \int_0^t \frac{1}{2} m \left(\frac{d\xi(s)}{ds} \right)^2 + V(\xi(s)) + m \frac{d^2 \xi(s)}{ds^2} \cdot \xi(s) ds$.

THEOREME 1 *The deterministic action is a solution to the deterministic Hamilton-Jacobi equations:*

$$\frac{\partial S(\mathbf{x}, t; \mathbf{x}_0, \mathbf{v}_0)}{\partial t} \Big|_{\mathbf{x}=\xi(t)} + \frac{1}{2m} (\nabla S(\mathbf{x}, t; \mathbf{x}_0, \mathbf{v}_0))^2 \Big|_{\mathbf{x}=\xi(t)} + V(\mathbf{x}) \Big|_{\mathbf{x}=\xi(t)} = 0 \quad (12)$$

$$\frac{d\xi(t)}{dt} = \frac{\nabla S(\mathbf{x}, t; \mathbf{x}_0, \mathbf{v}_0)}{m} \Big|_{\mathbf{x}=\xi(t)} \quad (13)$$

$$S(\mathbf{x}, 0; \mathbf{x}_0, \mathbf{v}_0) = m \mathbf{v}_0 \mathbf{x} \quad \text{and} \quad \xi(0) = \mathbf{x}_0. \quad (14)$$

The deterministic action satisfies the Hamilton-Jacobi equations only along the classical trajectory $\xi(t)$. It is the action introduced by Rybakov² for a soliton. We will interpret these equations in section 4 when we will study the QM-CM convergence.

III. INTERPRETATION OF THE EULER-LAGRANGE IN MINPLUS ANALYSIS

There exists a new branch of mathematics, the Minplus analysis, which studies nonlinear problems through a linear approach, cf. Maslov^{3,4} and Gondran^{5,6}. The idea is to substitute the usual scalar product $\int_X f(x)g(x)dx$ with the Minplus scalar product:

$$(f, g) = \inf_{x \in X} \{f(x) + g(x)\} \quad (15)$$

In the scalar product we replace the field of the real number $(R, +, \times)$ with the algebraic structure *Minplus* $(R \cup \{+\infty\}, \min, +)$, i.e. the set of real numbers (with the element infinity $\{+\infty\}$) endowed with the operation Min (minimum of two reals), which replaces

the usual addition, and with the operation $+$ (sum of two reals), which replaces the usual multiplication. The element $\{+\infty\}$ corresponds to the neutral element for the operation Min , $\text{Min}(\{+\infty\}, a) = a \ \forall a \in R$. This approach bears a close similarity to *the theory of distributions for the nonlinear case*; here, the operator is "linear" and continuous with respect to the Minplus structure, though *nonlinear* with respect to the classical structure $(R, +, \times)$. In this Minplus structure, the Hamilton-Jacobi equation is linear, because if $S_1(\mathbf{x}, t)$ and $S_2(\mathbf{x}, t)$ are solutions to (9), then $\min\{\lambda + S_1(\mathbf{x}, t), \mu + S_2(\mathbf{x}, t)\}$ is also a solution to the Hamilton-Jacobi equation (9).

The analog to the Dirac distribution $\delta(\mathbf{x})$ in Minplus analysis is the nonlinear distribution $\delta_{\min}(\mathbf{x}) = \{0 \text{ if } \mathbf{x} = \mathbf{0}, +\infty \text{ if not}\}$. With this nonlinear Dirac distribution, we can define elementary solutions as in classical distribution theory. In particular, we have:

The classical Euler-Lagrange action $S_{cl}(\mathbf{x}, t; \mathbf{x}_0)$ is the elementary solution to the Hamilton-Jacobi equations (9)(10) in the Minplus analysis with the initial condition

$$S(\mathbf{x}, 0) = \delta_{\min}(\mathbf{x} - \mathbf{x}_0) = \{0 \text{ if } \mathbf{x} = \mathbf{x}_0, +\infty \text{ if not}\}. \quad (16)$$

The Hamilton-Jacobi action $S(\mathbf{x}, t)$ is then given by the Minplus integral:

$$S(\mathbf{x}, t) = \inf_{\mathbf{x}_0} \{S_0(\mathbf{x}_0) + S_{cl}(\mathbf{x}, t; \mathbf{x}_0)\} \quad (17)$$

in analogy with the solution to the heat transfer equation given by the classical integral:

$$S(x, t) = \int S_0(x_0) \frac{1}{2\sqrt{\pi t}} e^{-\frac{(x-x_0)^2}{4t}} dx_0. \quad (18)$$

In this Minplus analysis, the Legendre-Fenchel transform is the analog to the Fourier transform. This transform is known to have many applications in physics: it sets the correspondence between the Lagrangian and the Hamiltonian of a physical system; it sets the correspondence between microscopic and macroscopic models; it is also at the basis of multifractal analysis relevant to modeling turbulence in fluid mechanics⁶.

IV. THE TWO LIMITS OF THE SCHRÖDINGER EQUATION IN THE SEMI-CLASSICAL APPROXIMATION

Let us consider the wave function solution to the Schrödinger equation $\Psi(\mathbf{x}, t)$:

$$i\hbar \frac{\partial \Psi}{\partial t} = -\frac{\hbar^2}{2m} \Delta \Psi + V(\mathbf{x}, t) \Psi \quad (19)$$

$$\Psi(\mathbf{x}, 0) = \Psi_0(\mathbf{x}). \quad (20)$$

With the variable change $\Psi(\mathbf{x}, t) = \sqrt{\rho^h(\mathbf{x}, t)} \exp(i \frac{S^h(\mathbf{x}, t)}{\hbar})$, the Schrödinger equation can be decomposed into Madelung equations⁷ (1926):

$$\frac{\partial S^h(\mathbf{x}, t)}{\partial t} + \frac{1}{2m}(\nabla S^h(\mathbf{x}, t))^2 + V(\mathbf{x}, t) - \frac{\hbar^2}{2m} \frac{\Delta \sqrt{\rho^h(\mathbf{x}, t)}}{\sqrt{\rho^h(\mathbf{x}, t)}} = 0 \quad (21)$$

$$\frac{\partial \rho^h(\mathbf{x}, t)}{\partial t} + \text{div}(\rho^h(\mathbf{x}, t) \frac{\nabla S^h(\mathbf{x}, t)}{m}) = 0 \quad (22)$$

with initial conditions

$$\rho^h(\mathbf{x}, 0) = \rho_0^h(\mathbf{x}) \quad \text{and} \quad S^h(\mathbf{x}, 0) = S_0^h(\mathbf{x}). \quad (23)$$

We consider two cases *depending on the preparation of the particles*^{8,9}.

Définition 1 - *The **statistical semi-classical case** where*

- *the initial probability density $\rho_0^h(\mathbf{x})$ and the initial action $S_0^h(\mathbf{x})$ are regular functions $\rho_0(\mathbf{x})$ and $S_0(\mathbf{x})$ not depending on \hbar .*
- *the interaction with the potential field $V(\mathbf{x}, t)$ can be described classically.*

It is the case of a set of non-interacting particles all prepared in the same way: a free particle beam in a linear potential, an electronic or C_{60} beam in the Young's slits diffraction, or an atomic beam in the Stern and Gerlach experiment.

Définition 2 - *The **determinist semi-classical case** where*

- *the initial probability density $\rho_0^h(\mathbf{x})$ converges, when $\hbar \rightarrow 0$, to a Dirac distribution and the initial action $S_0^h(\mathbf{x})$ is a regular function $S_0(\mathbf{x})$ not depending on \hbar .*
- *the interaction with the potential field $V(\mathbf{x}, t)$ can be described classically.*

This situation occurs when the wave packet corresponds to a quasi-classical coherent state, introduced in 1926 by Schrödinger¹⁰. The field quantum theory and the second quantification are built on these coherent states¹¹. The existence for the hydrogen atom of a localized wave packet whose motion is on the classical trajectory (an old dream of Schrödinger's) was predicted in 1994 by Bialynicki-Birula, Kalinski, Eberly, Buchleitner et Delande¹²⁻¹⁴, and discovered recently by Maeda and Gallagher¹⁵ on Rydberg atoms.

THEOREME 2 ^{8,9} *For particles in the statistical semi-classical case, the probability density $\rho^h(\mathbf{x}, t)$ and the action $S^h(\mathbf{x}, t)$, solutions to the Madelung equations (21)(22)(23), converge, when $\hbar \rightarrow 0$, to the classical density $\rho(\mathbf{x}, t)$ and the classical action $S(\mathbf{x}, t)$, solutions to the statistical Hamilton-Jacobi equations:*

$$\frac{\partial S(\mathbf{x}, t)}{\partial t} + \frac{1}{2m}(\nabla S(\mathbf{x}, t))^2 + V(\mathbf{x}, t) = 0 \quad (24)$$

$$\frac{\partial \rho(\mathbf{x}, t)}{\partial t} + \text{div} \left(\rho(\mathbf{x}, t) \frac{\nabla S(\mathbf{x}, t)}{m} \right) = 0 \quad \forall (\mathbf{x}, t) \quad (25)$$

$$\rho(\mathbf{x}, 0) = \rho_0(\mathbf{x}) \quad \text{and} \quad S(\mathbf{x}, 0) = S_0(\mathbf{x}). \quad (26)$$

We give some indications on the demonstration of this theorem and we propose its interpretation. Let us consider the case where the wave function $\Psi(\mathbf{x}, t)$ at time t is written as a function of the initial wave function $\Psi_0(\mathbf{x})$ by the Feynman paths integral formula¹⁶ (p. 58):

$$\Psi(\mathbf{x}, t) = \int F(t, \hbar) \exp\left(\frac{i}{\hbar} S_{cl}(\mathbf{x}, t; \mathbf{x}_0)\right) \Psi_0(\mathbf{x}_0) d\mathbf{x}_0$$

where $F(t, \hbar)$ is an independent function of \mathbf{x} and of \mathbf{x}_0 and where $S_{cl}(\mathbf{x}, t; \mathbf{x}_0)$ is the classical action. In the statistical semi-classical case, the wave function is written $\Psi(\mathbf{x}, t) = F(t, \hbar) \int \sqrt{\rho_0(\mathbf{x}_0)} \exp\left(\frac{i}{\hbar} (S_0(\mathbf{x}_0) + S_{cl}(\mathbf{x}, t; \mathbf{x}_0))\right) d\mathbf{x}_0$. The theorem of the stationary phase shows that, if \hbar tends towards 0, we have $\Psi(\mathbf{x}, t) \sim \exp\left(\frac{i}{\hbar} \min_{\mathbf{x}_0} (S_0(\mathbf{x}_0) + S_{cl}(\mathbf{x}, t; \mathbf{x}_0))\right)$, that is to say that the quantum action $S^h(\mathbf{x}, t)$ converges to the function

$$S(\mathbf{x}, t) = \min_{\mathbf{x}_0} (S_0(\mathbf{x}_0) + S_{cl}(\mathbf{x}, t; \mathbf{x}_0)) \quad (27)$$

which is the solution to the Hamilton-Jacobi equation (9) with the initial condition (10). Moreover, as the quantum density $\rho^h(\mathbf{x}, t)$ satisfies the continuity equation (22), we deduce, since $S^h(\mathbf{x}, t)$ tends towards $S(\mathbf{x}, t)$, that $\rho^h(\mathbf{x}, t)$ converges to the classical density $\rho(\mathbf{x}, t)$, which satisfies the continuity equation (25). We obtain both announced convergences.

The statistical Hamilton-Jacobi equations correspond to a set of independent classical particles, in a potential field $V(\mathbf{x}, t)$, and for which we only know at the initial time the probability density $\rho_0(\mathbf{x})$ and the velocity $\mathbf{v}(\mathbf{x}) = \frac{\nabla S_0(\mathbf{x}, t)}{m}$.

Définition 3 - *N identical particles, prepared in the same way, with the same initial density $\rho_0(\mathbf{x})$, the same initial action $S_0(\mathbf{x})$, and evolving in the same potential $V(\mathbf{x}, t)$ are called non-discerned.*

We refer to these particles as non-discerned and not as indistinguishable because, if their initial positions are known, their trajectories will also be known. Nevertheless, when one counts them, they will have the same properties as the indistinguishable ones. Thus, if the initial density $\rho_0(\mathbf{x})$ is given, and one randomly chooses N particles, the $N!$ permutations are strictly equivalent and do not correspond to the same configuration as for indistinguishable particles. This indistinguishability of classical particles provides a very simple and natural explanation to the Gibbs paradox.

In the statistical semi-classical case, the uncertainty about the position of a quantum particle corresponds to an uncertainty about the position of a classical particle, whose initial density alone has been defined. *In classical mechanics, this uncertainty is removed by giving the initial position of the particle. It would be illogical not to do the same in quantum mechanics.* We assume that for *the statistical semi-classical case*, a quantum particle is not well described by its wave function. One therefore needs to add its initial position and it follows that we introduce the so-called de Broglie-Bohm trajectories^{17,18} with the velocity $\mathbf{v}^h(\mathbf{x}, t) = \frac{1}{m} \nabla S^h(\mathbf{x}, t)$.

The convergence study of the determinist semi-classical case is mathematically very difficult. We only study the example of a coherent state where an explicit calculation is possible.

For the two dimensional harmonic oscillator, $V(\mathbf{x}) = \frac{1}{2}m\omega^2\mathbf{x}^2$, coherent states are built²⁴ from the initial wave function $\Psi_0(\mathbf{x})$ which corresponds to the density and initial action $\rho_0^h(\mathbf{x}) = (2\pi\sigma_h^2)^{-1} e^{-\frac{(\mathbf{x}-\mathbf{x}_0)^2}{2\sigma_h^2}}$ and $S_0(\mathbf{x}) = S_0^h(\mathbf{x}) = m\mathbf{v}_0 \cdot \mathbf{x}$ with $\sigma_h = \sqrt{\frac{\hbar}{2m\omega}}$. Here, \mathbf{v}_0 and \mathbf{x}_0 are still constant vectors and independent from \hbar , but σ_h will tend to 0 as \hbar . With initial conditions, the density $\rho^h(\mathbf{x}, t)$ and the action $S^h(\mathbf{x}, t)$, solutions to the Madelung equations (21)(22)(23), are equal to²⁴: $\rho^h(\mathbf{x}, t) = (2\pi\sigma_h^2)^{-1} e^{-\frac{(\mathbf{x}-\xi(t))^2}{2\sigma_h^2}}$ and $S^h(\mathbf{x}, t) = +m\frac{d\xi(t)}{dt} \cdot \mathbf{x} + g(t) - \hbar\omega t$, where $\xi(t)$ is the trajectory of a classical particle evolving in the potential $V(\mathbf{x}) = \frac{1}{2}m\omega^2\mathbf{x}^2$, with \mathbf{x}_0 and \mathbf{v}_0 as initial position and velocity and $g(t) = \int_0^t (-\frac{1}{2}m(\frac{d\xi(s)}{ds})^2 + \frac{1}{2}m\omega^2\xi(s)^2)ds$.

THEOREME 3 ^{8,9}- When $\hbar \rightarrow 0$, the density $\rho^h(\mathbf{x}, t)$ and the action $S^h(\mathbf{x}, t)$ converge to

$$\rho(\mathbf{x}, t) = \delta(\mathbf{x} - \xi(t)) \quad \text{and} \quad S(\mathbf{x}, t) = m\frac{d\xi(t)}{dt} \cdot \mathbf{x} + g(t) \quad (28)$$

where $S(\mathbf{x}, t)$ and the trajectory $\xi(t)$ are solutions to the determinist Hamilton-Jacobi equations:

$$\frac{\partial S(\mathbf{x}, t)}{\partial t} \Big|_{\mathbf{x}=\xi(t)} + \frac{1}{2m}(\nabla S(\mathbf{x}, t))^2 \Big|_{\mathbf{x}=\xi(t)} + V(\mathbf{x}) \Big|_{\mathbf{x}=\xi(t)} = 0 \quad (29)$$

$$\frac{d\xi(t)}{dt} = \frac{\nabla S(\xi(t), t)}{m} \quad (30)$$

$$S(\mathbf{x}, 0) = m\mathbf{v}_0 \cdot \mathbf{x} \quad \text{and} \quad \xi(0) = \mathbf{x}_0. \quad (31)$$

Therefore, the kinematic of the wave packet converges to the single harmonic oscillator described by $\xi(t)$. Because this classical particle is completely defined by its initial conditions \mathbf{x}_0 and \mathbf{v}_0 , it can be considered as a *discerned particle*. It is then possible to consider, unlike in the statistical semi-classical case, that the wave function can be viewed as a single quantum particle. The *determinist semi-classical case* is in line with the Copenhagen interpretation of the wave function, which contains all the information on the particle. A natural interpretation is proposed by Schrödinger¹⁰ in 1926 for the coherent states of the harmonic oscillator: the quantum particle is a spatially extended particle, represented by a wave packet whose center follows the classical trajectory.

V. THE NON SEMI-CLASSICAL CASE

The Broglie-Bohm and Schrödinger interpretations correspond to the semi-classical approximation. They correspond to the two interpretations proposed in 1927 at the Solvay congress by de Broglie and Schrödinger. The principle of an interpretation that depends on the particle preparation conditions is not really new. It had already been figured out by Einstein and de Broglie. For Louis de Broglie, its real interpretation was the double solution theory introduced in 1927 in which the pilot-wave is just a low-level product²⁵:

"I introduced as a 'double solution theory' the idea that it was necessary to distinguish two different solutions but both linked to the wave equation, one that I called wave u which was a real physical wave but not normalizable having a local anomaly defining the particle and represented by a singularity, the other one as the Schrödinger Ψ wave, which is normalizable without singularities and being a probability representation."

We consider as interesting L. de Broglie's idea of the existence of a statistical wave, Ψ and of a soliton wave u ; however, it is not a double solution that appears here but a double interpretation of the wave function according to the initial conditions.

Einstein's point of view is well summed up in one of his final papers (1953), "*Elementary reflections concerning the foundation of quantum mechanics*" in homage to Max Born²⁶:

"The fact that the Schrödinger equation associated with the Born interpretation does not

lead to a description of the "real states" of an individual system, naturally incites one to find a theory that is not subject to this limitation. Up to now, the two attempts have in common that they conserve the Schrödinger equation and abandon the Born interpretation. The first one, which marks de Broglie's comeback, was continued by Bohm.... The second one, which aimed to get a "real description" of an individual system and which might be based on the Schrödinger equation is very late and is from Schrödinger himself. The general idea is briefly the following : the function ψ represents in itself the reality and it is not necessary to add it to Born's statistical interpretation.[...] From previous considerations, it results that the only acceptable interpretation of the Schrödinger equation is the statistical interpretation given by Born. Nevertheless, this interpretation doesn't give the "real description" of an individual system, it just gives statistical statements of entire systems."

Thus, it is because de Broglie and Schrödinger maintain the Schrödinger equation that Einstein, who considers it as fundamentally statistical, rejected each of their interpretations. Einstein thought that it was not possible to obtain an individual deterministic behavior from the Schrödinger equation. It is the same for Heisenberg who developed matrix mechanics and the second quantization from the example of transitions in a hydrogen atom.

But there exist situations in which the semi-classical approximation is not valid. It is in particular the case of state transitions in a hydrogen atom. Indeed, since Delmelt's experiment²³ in 1986, the physical reality of individual quantum jumps has been fully validated. The semi-classical approximation, where the interaction with the potential field can be described classically, is no longer possible and it is necessary to quantify the electromagnetic field since the exchanges occur photon by photon. In this situation, the Schrödinger equation cannot give a deterministic interpretation and the statistical Born interpretation seems to be the only valid one. It was the third interpretation proposed in 1927 at the Solvay congress, the interpretation that was recognized as the right one in spite of Einstein's, de Broglie's and Schrödinger's criticisms.

This doesn't mean that it is necessary to abandon determinism and realism in quantum mechanics, but rather that the Schrödinger wave function doesn't allow, in this case, to obtain an individual behavior of a particle. An individual interpretation needs to use the creation and annihilation operators of the quantum Field Theory, but this interpretation still remains statistical.

We hypothesize that it is possible to construct a deterministic quantum field theory that

extends to the non semi-classical interpretation of the double semi-classical interpretation. First, as shown by de Muynck²⁷, we can construct a theory with discerned (labeled) creation and annihilation operators in addition to the usual non-discerned creation and annihilation operators. But, to satisfy the determinism, it is necessary to search, at lower scale, the mechanisms that allow the emergence of the creation operator.

VI. CONCLUSION

The study of the convergence of the Madelung equations when $\hbar \rightarrow 0$, gives the following results:

- In the statistical semi-classical case, the quantum particles converge to classical non-discerned ones satisfying the statistical Hamilton-Jacobi equations, and the Broglie-Bohm pilot-wave interpretation is relevant.

- In the determinist semi-classical case, the quantum particles converge to classical discerned ones satisfying the determinist Hamilton-Jacobi equations. And we can make a realistic and deterministic assumption such as the Schrödinger interpretation.

This double interpretation seems to be the interpretation of Louis de Broglie's "double solution" idea.

- In the case where the semi-classical approximation is no longer valid, as in the transition states in the hydrogen atom, Louis de Broglie's "double solution" is not directly applicable. But, we hypothesize that it is possible to construct a deterministic quantum field theory that extends this double interpretation to the non semi-classical case.

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¹ L. C. Evans, *Partial Differential Equations*, Graduate Studies in Mathematics 19, American Mathematical Society, 1998, p.123-124.

² Yu.S. Rybakov, in *Proceeding of the first International Conference on Theoretical Physics* (Moscow, 2011), p.155.

³ V.P. Maslov and S.N. Samborski, *Idempotent Analysis*, Advances in Soviet Mathematics, 13, American Math Society, Providence (1992).

- ⁴ V.N. Kolokoltsov and V.P. Maslov, *Idempotent Analysis and its applications*, Klumer Acad. Publ., 1997.
- ⁵ M. Gondran, "Analyse MinPlus", C. R. Acad. Sci. Paris **323**, 371-375 (1996).
- ⁶ M. Gondran et M. Minoux, *Graphs, Dioïds and Semi-rings: New models and Algorithms*, Springer, Operations Research/Computer Science Interfaces, 2008, chap.7.
- ⁷ E. Madelung, "Quantentheorie in hydrodynamischer Form", Zeit. Phys. **40**, 322-6 (1926).
- ⁸ M. Gondran and A. Gondran, "Discerned and non-discerned particles in classical mechanics and convergence of quantum mechanics to classical mechanics", Annales de la Fondation Louis de Broglie, vol. 36, 117-135 (2011).
- ⁹ M. Gondran and A. Gondran, "The two limits of the Schrödinger equation in the semi-classical approximation", Proceeding of AIP, Conference Foundations of Probability and Physics 6, Växjö, Sweden, june 2011, vol 1424, 2012.
- ¹⁰ E. Schrödinger, Der stetige bergang von der Mikro-zur Makromechanik, Naturwissenschaften **14**, 664-666 (1926).
- ¹¹ R. J. Glauber, in *Quantum Optics and Electronics*, Les Houches Lectures 1964, C. deWitt, A. Blandin and C. Cohen-Tanoudji eds., Gordon and Breach, New York, 1965.
- ¹² I. Bialynicki-Birula, M. Kalinski, and J. H. Eberly, Phys. Rev. Lett. **73**, 1777 (1994).
- ¹³ A. Buchleitner and D. Delande, Phys. Rev. Lett. **75**, 1487 (1995).
- ¹⁴ A. Buchleitner, D. Delande and J. Zakrzewski, "Non-dispersive wave packets in periodically driven quantum systems," Physics Reports **368** 409-547 (2002).
- ¹⁵ H. Maeda and T. F. Gallagher, Non dispersing Wave Packets, Phys. Rev. Lett. **92**, 133004-1 (2004).
- ¹⁶ R. Feynman and A. Hibbs, *Quantum Mechanics and Integrals*, McGraw-Hill, 1965.
- ¹⁷ L. de Broglie, J. de Phys. **8**, 225-241 (1927).
- ¹⁸ D. Bohm, "A suggested interpretation of the quantum theory in terms of "hidden" variables," Phys. Rev., **85**, 166-193 (1952).
- ¹⁹ C. Jönsson, "Elektroneninterferenzen an mehreren künstlich hergestellten Feinspalten," Z. Phys. **161**, 454-474 (1961), English translation "Electron diffraction at multiple slits," Am. J. Phys. **42**, 4-11 (1974).
- ²⁰ M. Gondran, and A. Gondran, "Numerical simulation of the double-slit interference with ultra-cold atoms", *Am. J. Phys.* **73**, 507-515 (2005).

- ²¹ D. Bohm, B.J. Hiley, *The Undivided Universe* (Routledge, London and New York, 1993).
- ²² P.R. Holland , *The quantum Theory of Motion*, Cambridge University Press, 1993.
- ²³ W. Nagourney, J. Sandberg, and H. Dehmelt, "Shelved optical electron amplifier: Observation of quantum jumps," *Phys. Rev. Lett.* **56**, 2797-2799 (1986).
- ²⁴ C. Cohen-Tannoudji, B. Diu, F. Laloë, *Quantum Mechanics*, Wiley, New York (1977).
- ²⁵ L. de Broglie, J.L. Andrade e Silva, *La Réinterprétation de la mécanique ondulatoire*, Gauthier-Villars (1971).
- ²⁶ A. Einstein, " Elementary Reflexion on Interpreting the Foundations of Quantum Mechanics ", in *Scientific Papers presented to Max Born*, Edimbourg, Olivier and Boyd, 1953
- ²⁷ W.M. de Muynck, "Distinguishable-and Indistinguishable-Particle; Descriptions of Systems of Identical Particles", *International Journal of Theoretical Physics* 14, n° 5, 327-346 (1975).